Forced small-amplitude water waves: a comparison of theory and experiment

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This paper describes an attempt to verify experimentally the wavemaker theory for a piston-type wavemaker. The theory is based upon the usual assumptions of classical hydrodynamics, i.e. that the fluid is inviscid, of uniform density, that motion starts from rest, and that non-linear terms are neglected. If the water depth, wavelength, wave period, and wavemaker stroke (of a harmonically oscillating wavemaker) are known, then the wavemaker theory predicts the wave motion everywhere, and in particular the wave height a few depths away from the wavemaker.

The experiments were conducted in a 100 ft. wave channel, and the wave-height envelope was measured with a combination hook-and-point gauge. A plane beach (sloping 1:15) to absorb the wave energy was located at the far end of the channel. The amplitude-reflexion coefficient was usually less than 10 %. Unless this reflexion effect is corrected for, it imposes one of the most serious limitations upon experimental accuracy. In the analysis of the present set of measurements, the reflexion effect is taken into account.

The first series of tests was concerned with verifying the wavemaker theory for waves of small steepness ($0{\cdot}002 \leqslant H/L \leqslant 0{\cdot}03$). For this range of wave steepnesses, the measured wave heights were found to be on the average 3·4 % below the height predicted by theory. The experimental error, as measured by the scatter about a line 3·4 % below the theory, was of the order of 3 %. The systematic deviation of 3·4 % is believed to be partly due to finite-amplitude effects and possibly to imperfections in the wavemaker motion.

The second series of tests was concerned with determining the effects of finite amplitude. For the range of wave steepnesses $0{\cdot}045 \leqslant H/L \leqslant 0{\cdot}048$, the measured wave heights were found to be on the average 10% below the heights predicted from the small-amplitude theory. The experimental error was again of the order of 3 %.

It is considered that these measurements confirm the validity of the small-amplitude wave theory. No confirmation of this accuracy has hitherto been given for forced motions.

1. Introduction

Some years ago, a group of engineers working at the Laboratoire Neyrpic at Grenoble published an account of a simple experiment (Neyrpic 1952) which gave results incompatible with the predictions of the mathematical theory of water waves, and which suggested that this theory was physically inadequate or even irrelevant. Even before this, the evidence for the theory had appeared incomplete and unsatisfactory. In the Neyrpic experiment, a paddle wavemaker (i.e. a plate hinged at the bottom of a wave channel) was given a small harmonic motion about a vertical mean position, and the wave height at some distance down the channel was measured; this was then compared with the theory of Havelock (1929), who had calculated the motion on the assumptions of classical hydrodynamics (see §§2, 3 below). It was found that the measured wave height was consistently about 30 % below the theoretical wave height. The assumptions of the theory seemed to be satisfied to a close approximation (see the papers by Suquet (1951) and Biésel & Suquet (1951) of the Laboratoire Neyrpic). The Neyrpic group was unable to make a more detailed study, and no explanation has since been given of the discrepancy.

Another attempted verification of theoretical wave-height predictions has been made by Cooper & Longuet-Higgins (1951), who investigated reflexion from a partially immersed vertical barrier by measuring the unsteady state before the secondary incident wave (see §4.1 below) had travelled back from the wavemaker. This unsteady state persisted for so many periods that it could be treated as a steady state with little loss of accuracy. The measured reflexion coefficient was always lower than the theoretical; the discrepancy tended to decrease as the depth of immersion of the lower edge of the barrier was increased, but was still of the order of 10 % when the depth of immersion was one-third of the wavelength. It was suggested that part of this energy loss occurred near the lower edge of the barrier where an eddying motion with separation was observed.

On the other hand, many experiments on particle orbits, frequencies, and velocities have verified the theoretical predictions (see also §2 below); in these experiments the orbits were correlated only with the local *measured* wave height. The suggestion might be made on the evidence of the Neyrpic experiments that the theory predicts correctly everything except wave heights (and presumably forces), to which a reduction factor of 0·7 to 0·9 should be applied. By what physical mechanism can a large part of the energy input be lost? Is the same correction factor required in all engineering applications? These and similar questions seemed sufficiently important to warrant an experimental study, which is described in the present paper.

The present experiment, like the Neyrpic experiment, is concerned with the investigation of a wavemaker; however, the results of this investigation agree well with the theoretical predictions of wave heights. We are unable to suggest a convincing reason for the discrepancy found by the Neyrpic group. The present experiment provides some physical verification for the wave-height predictions of the mathematical theory.

2. Small-amplitude waves: theory and experiment

The greater part of the mathematical theory of water waves is based on the following assumptions:

(1) Density variations and viscosity in the fluid are neglected. Then the motion, if originally started from rest, is irrotational and can be described by a velocity potential.

(2) Non-linear terms in the equations of motion are neglected; this seems reasonable if the amplitude of motion is sufficiently small. The mathematical difficulties are greatly increased when non-linear or viscous effects are included, and comparatively little is known about these.

Accounts of the linearized theory are given in most text-books on hydrodynamics (e.g. see Lamb 1932, Chapters 8 and 9). Recent developments are described by Stoker (1957). Many experiments have been made to check the linearized theory, but (as already noted in § 1) these are concerned with regular wave trains or with frequencies and velocities rather than with wave heights or forces. To mention just a few investigations, wave velocities and particle orbits have been measured by the Beach Erosion Board (1941) and more accurately by Suquet & Wallet (1953), and were found to agree with theory within the experimental error of a few per cent. Suquet & Wallet found that the wave profiles agreed well with the non-linear theory. The mean drift velocity of particles (mass transport), however, differed considerably in some shallow-water experiments. This discrepancy has been observed in detail (Bagnold 1947) and explained as a consequence of viscosity (Longuet-Higgins 1953); it is, however, a non-linear effect. The theory has also been confirmed for frequencies, e.g. for resonance frequencies of edge waves by Ursell (1952) and for stability under vertical oscillatory accelerations where a curve of neutral stability was successfully predicted by Benjamin & Ursell (1954). The discrepancy was within the experimental error of a few per cent. The frequency of free oscillation in a vessel has been verified to within a few per cent by Case & Parkinson (1957). On a much larger scale, the group velocity of ocean swell has been found to agree with theory by Barber & Ursell (1948).

In the theory, viscosity has been neglected; this is not always realistic, particularly when flow separation takes place. However, similar difficulties arise in model experiments as well as in the theory. To fix ideas, let us consider a floating body placed in waves. What is the interaction between the body and the waves? What are the forces on the body? Such questions are conventionally treated by a model experiment in the laboratory where forces and velocities can be measured. The problem is, how to derive from these the full-scale forces and velocities. For it is only when viscosity can be neglected that simple Froude scaling is appropriate. Sometimes more complex scaling procedures are used, e.g. in the measurement of ship drag it is customary to separate the force somewhat arbitrarily into viscous and wave-drag components which are scaled according to the Reynolds and Froude laws respectively. By this method, full-scale and model tests are found to correlate quite reasonably. A similar procedure has been used to analyse wave forces on vertical fixed piles. The agreement is best when viscous effects are small.

In addition to these difficulties of principle, model tests suffer from experimental errors which can be large. There are therefore definite advantages in studying problems theoretically, and to try to find at least the non-viscous effects from the mathematical theory; this is perhaps as accurate as the traditional experimental method. But before the results of the theory can be used with any confidence, the relevance of the theory must be established by careful experiments; in particular, the Neyrpic experiment, which seems to be incompatible with the theory, requires further study.

3. Wavemaker theory

The Neyrpic experiment was carried out with a paddle wavemaker, while the present study was made with a piston wavemaker. The relevant theory has long been known (Havelock 1929), and in its general form applies to both of these and to a much wider class of wavemakers. The general form of the wavemaker theory will be presented first, from which the special cases of piston and paddle wavemakers are obtained by substituting the appropriate boundary conditions.

Let us consider two-dimensional small-amplitude waves, which are generated in a semi-infinite channel ($0 \leqslant x < \infty$, $0 \leqslant y \leqslant h$) of constant depth h by giving a simple-harmonic motion to a moving partition oscillating about $x = 0$; the co-ordinate y is chosen to increase with depth. The amplitude of motion of the partition is assumed to be so small that the equations can be linearized, and the partition is assumed to remain nearly vertical. To this approximation, the horizontal fluid velocity on $x = 0$ is equal to the horizontal component of the velocity of the partition.

Viscosity and surface tension are neglected, and the simple-harmonic wave motion is described by a velocity potential $\phi(x,y,t)$ which satisfies (see Lamb 1932, §227)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{in the region} \quad 0 \leqslant x < \infty, \quad 0 \leqslant y \leqslant h, \tag{3.1}$$

with the boundary conditions

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = h, \tag{3.2}$$

$$\sigma^2\phi + g\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{3.3}$$

$$\frac{\partial\phi}{\partial x} = U(y)\sin\sigma t \quad \text{on} \quad x = 0, \tag{3.4}$$

and also the following boundary condition at infinity:

as $x \to \infty$, $\phi(x,y,t) \to$ *a progressive wave travelling in the positive x-direction.* (3.5)

Here $2\pi/\sigma$ is the period, g is the acceleration of gravity and $U(y)$ is the prescribed horizontal velocity on the wavemaker. The condition (3.3) expresses the constancy of pressure on the free surface; the *radiation condition* (3.5) is valid in a semi-infinite channel, or in a finite channel when there is complete absorption by the beach, but is not exactly satisfied when the absorption is incomplete. According to Havelock, the potential is of the form:

$$\phi(x,y,t) = A_0 \cosh k_0(h-y)\cos(k_0x-\sigma t) + \sin\sigma t \sum_{n=1}^{\infty} A_n e^{-k_n x}\cos k_n(h-y), \tag{3.6}$$

where k_0 is the real positive root of $\sigma^2 = gk\tanh kh$, and $k_1, k_2, \ldots$, are the real positive roots of $\sigma^2 = -gk\tan kh$. It can be shown that $(n-\frac{1}{2})\pi < k_n h < n\pi$, whence $k_n h > \frac{1}{2}\pi$ for all n, and so the terms under the summation in (3.6) are negligible at a distance of about $3h$ from the wavemaker. The constants $A_0, A_1, A_2, \ldots$ are to be chosen to satisfy equation (3.4); thus, from (3.6), it follows that

$$U(y) = A_0 k_0 \cosh k_0(h-y) - \sum_{n=1}^{\infty} A_n k_n \cos k_n(h-y), \tag{3.7}$$

where the functions on the right of (3.7) form a complete orthogonal set, and so

$$\int_0^h U(y)\cosh k_0(h-y)\,dy = A_0 k_0 \int_0^h \cosh^2 k_0(h-y)\,dy,$$

$$\int_0^h U(y)\cos k_n(h-y)\,dy = -A_n k_n \int_0^h \cos^2 k_n(h-y)\,dy.$$

In particular, the wave profile at infinity is (Lamb, § 227)

$$\frac{1}{g}\left(\frac{\partial\phi}{\partial t}\right)_{y=0} = \frac{\sigma A_0}{g}\cosh k_0 h \sin(k_0 x - \sigma t) = \frac{\sigma}{gk_0}\cosh k_0 h \sin(k_0x-\sigma t)\frac{\int_0^h U(y)\cosh k_0(h-y)\,dy}{\int_0^h \cosh^2 k_0(h-y)\,dy}. \tag{3.8}$$

For a *piston wavemaker*, $U(y) = \text{constant} = \frac{1}{2}S\sigma$, where S is the stroke of the wavemaker, and thus

$$\frac{H}{S} = \frac{\text{wave height}}{\text{wavemaker stroke}} = \frac{\sigma^2\cosh k_0 h}{gk_0}\frac{\int_0^h \cosh k_0(h-y)\,dy}{\int_0^h \cosh^2 k_0(h-y)\,dy} = \frac{2(\cosh 2k_0h - 1)}{\sinh 2k_0 h + 2k_0 h}, \tag{3.9}$$

since $\sigma^2 = gk_0 \tanh k_0 h$. In these expressions H is the wave height at a distance (say $x > 3h$) from the wavemaker, and $2\pi/k_0$ is the wave length at a distance from the wavemaker.

For a *paddle wavemaker* hinged at the bottom, we have $U(y) = \frac{1}{2}S\sigma\{1-(y/h)\}$, and hence

$$\frac{H}{S} = \frac{\sigma^2\cosh k_0 h}{gk_0}\frac{\int_0^h \{1-(y/h)\}\cosh k_0(h-y)\,dy}{\int_0^h \cosh^2 k_0(h-y)\,dy} = \frac{4\sinh k_0 h}{k_0 h}\,\frac{k_0 h\sinh k_0 h - \cosh k_0 h + 1}{\sinh 2k_0 h + 2k_0 h}. \tag{3.10}$$

Equation (3.10) has been derived and discussed by Biésel & Suquet (1951) and by Suquet (1951) and tested experimentally by the Neyrpic engineers. The present investigation was made on the piston wavemaker (for which equation (3.9) is appropriate) at the Hydrodynamics Laboratory of the Massachusetts Institute of Technology.

4. Effects not included in the simple theory

4.1. *Reflexion from the beach*

In § 3 it was assumed that the wave channel is infinitely long, or that the absorption of wave energy is complete. In practice it is incomplete (the amplitude reflexion coefficient is at least a few per cent., e.g. see Greslou & Mahe (1954)), and unless it is corrected for, it is one of the most serious limitations on experimental accuracy. Let us consider how the steady state is initially set up. When the wavemaker is first set in motion, a progressive wave train (primary incident wave) travels towards the beach. There it is partially reflected (primary reflected wave); the reflected amplitude is usually only a small fraction of the incident amplitude. The primary reflected wave is reflected (almost completely) from the wavemaker, as a secondary incident wave; this is reflected from the beach as a secondary reflected wave; and so on. The higher reflexions have progressively smaller amplitudes, and the steady state is almost attained after a few reflexions. Thus, if the reflexion is 10 % or less, the secondary reflected wave is at most 1 % of the primary incident wave, and is usually negligible. This description in terms of wave trains is valid only at a distance from the wavemaker and the beach (see equation (3.6) for the end effect near the wavemaker); thus, in this middle region, the motion consists of an incident wave train (the sum of the primary, secondary, etc., incident waves) and a reflected wave train (the sum of the primary, secondary, etc., reflected waves). However, the relative phases of primary, secondary and higher-order waves depend on the effective length of the channel and are unknown. It will be noted that the final incident wave is not given by the theory of § 3, which actually predicts the primary incident wave. *It is the* primary wave which must be obtained from the measurements. If the radiation condition is dropped and a standing wave now included, the potential valid except near the beach and satisfying all conditions of § 3 except (3.5) is

$$\phi(x,y,t) = A_0 \cosh k_0(h-y)\,[\cos(k_0x-\sigma t)+2\epsilon\cos k_0x\cos(\sigma t+\delta)] + \sin\sigma t\sum_{n=1}^{\infty} A_n e^{-k_n x}\cos k_n(h-y) + \textit{negligible end effects from the beach}, \tag{4.1}$$

where the coefficients A_n $(n = 0, 1, \ldots)$ are the same as in § 3, and ϵ and δ are unknown parameters depending on beach characteristics and length of channel.

The wave profile is (in an obvious notation) of the form

$$\sin(k_0x-\sigma t)-2\alpha\epsilon\cos k_0x\sin(\sigma t+\delta). \tag{4.2}$$

This involves the assumption that at the wavemaker the reflexion is complete. From the description at the beginning of this section it is easily seen that ϵ is nearly equal to the reflexion coefficient when ϵ is small. (The reflexion coefficient is by definition the amplitude ratio of the primary reflected wave to the primary incident wave, and also of the total reflected wave to the total incident wave.) The amplitude α of (4.2) is to be compared with the half-stroke $\frac{1}{2}H$ given by (3.9). It will now be shown how α can be derived from measurements of the wave profile. An obvious method is to measure the wave height at points where $\cos k_0 x = 0$, i.e. where $x = (\frac{1}{2}m + \frac{1}{4})$ wavelengths from the wavemaker (and not too near to the wavemaker or the beach). It has been found that this method does not lead to very consistent results, perhaps because of slight irregularities in the motion of the wavemaker, and another method of analysis has been used which depends on mean values rather than on values at one point. The amplitude of oscillation a at a distance x from the wavemaker is seen (from (4.2)) to be given by

$$a^2 = \alpha^2[1 + 2\epsilon\cos\delta + 2\epsilon\cos(2k_0x+\delta) + 2\epsilon^2\cos 2k_0x + 2\epsilon^2],$$

and when ϵ is small

$$a = \alpha[1 + \epsilon\cos(2k_0x+\delta) + \epsilon\cos\delta + O(\epsilon^2)]. \tag{4.3}$$

Thus the wave amplitude varies sinusoidally from point to point, between $a_{\max}$ and $a_{\min}$, where

$$a_{\max} = (1+\epsilon+\epsilon\cos\delta),$$
$$a_{\min} = (1-\epsilon+\epsilon\cos\delta).$$

It is found that the reflexion coefficient ϵ_r is given by

$$\epsilon_r = \frac{a_{\max}-a_{\min}}{a_{\max}+a_{\min}} = \frac{\epsilon}{1+\epsilon\cos\delta} = \epsilon + O(\epsilon^2). \tag{4.4}$$

Equation (4.3) predicts that the sinusoidal variation of amplitude along the channel has a wavelength equal to half the wavelength of the incident wave; if measurements show such a variation, then ϵ_r can be obtained. The averaged position of the maxima and minima gives an estimate for δ. Once ϵ (assuming $\epsilon \doteqdot \epsilon_r$) and δ are known, an estimate for α is found by averaging the amplitude over half a wavelength. From equation (4.3), we have

$$a_{\mathrm{av}} = \alpha[1 + \epsilon\cos\delta + O(\epsilon^2)], \tag{4.5}$$

whence

$$\alpha = \frac{a_{\mathrm{av}}}{1+\epsilon\cos\delta} \tag{4.6}$$

or, less accurately,

$$\alpha = a_{\mathrm{av}} \quad \text{if} \quad \epsilon \text{ is small.} \tag{4.7}$$

It is the result calculated from equation (4.6) which is to be compared with the theoretical prediction (equation (3.9)) with $H = 2\alpha$.

4.2. *Attenuation*

The analysis in terms of an incident and a standing wave (equation (4.2)) is not strictly appropriate when viscosity is taken into account. Viscosity causes a slight attenuation of the wave height as the incident wave travels from the wavemaker towards the beach and as the reflected wave travels from the beach towards the wavemaker. The theoretical estimate is due to Hunt (1952) who calculated the dissipation of energy in the boundary layers on the side-walls and on the bottom of the channel, making plausible assumptions. It would be expected that boundary layers account for most of the dissipation. If the amplitude attenuation is adequately described by a damping factor $\exp(-K'x)$, it is found that, for a progressive wave train,

$$K' = \frac{2k_0}{b}\left(\frac{\nu}{2\sigma}\right)^{\frac{1}{2}}\left(\frac{k_0 b + \sinh 2k_0 h}{2k_0 h + \sinh 2k_0 h}\right), \tag{4.8}$$

where b is the width of the channel, ν is the kinematic viscosity, and the other symbols are defined in § 3. This leads to values of K' of the order of $5 \times 10^{-4}\,\text{ft.}^{-1}$, for which this effect is just measurable in the M.I.T. wave channel.

4.3. *Higher harmonics*

The motion of the wavemaker is not quite simple harmonic; the second harmonic may be of the order of 5 %, too large to be neglected. However, the third and higher harmonics will be assumed to be negligible. The motion at a fixed point is then approximately of the form

$$f(t,\epsilon') = \alpha' \cos\sigma t + \alpha'\epsilon' \cos(2\sigma t + \delta'), \tag{4.9}$$

where α' is the local amplitude that is to be found, δ' is a constant but unknown phase angle, and ϵ' is a number much smaller than unity (ϵ'^2 is negligible when compared with unity). It will be shown that the difference between the maximum and minimum values of f is

$$f_{\max} - f_{\min} = 2\alpha'\{1 + O(\epsilon')^2\}, \tag{4.10}$$

involving only second and higher powers of ϵ'. Therefore this measurement is sufficient for our purpose.

To prove equation (4.10), denote by $t_{\max}(\epsilon')$ and $t_{\min}(\epsilon')$ the times at which $f(t,\epsilon')$ attains its maximum and minimum, respectively. These times depend on ϵ'. Clearly in any one cycle $t_{\max}(0) - t_{\min}(0) = \pi/\sigma$. Consider

$$[f\{t_{\max}(\epsilon'),\epsilon'\} - f\{t_{\min}(\epsilon'),\epsilon'\}].$$

This is a function of ϵ' which can be expanded in powers of ϵ'. The coefficient of ϵ' is

$$\frac{\partial f}{\partial t}(t_{\max},\epsilon')\frac{dt_{\max}}{d\epsilon'} + \frac{\partial f}{\partial \epsilon'}(t_{\max},\epsilon') - \frac{\partial f}{\partial t}(t_{\min},\epsilon')\frac{dt_{\min}}{d\epsilon'} - \frac{\partial f}{\partial \epsilon'}(t_{\min},\epsilon')$$

evaluated at $\epsilon' = 0$. Clearly

$$\frac{\partial f}{\partial t}(t_{\max},\epsilon') = \frac{\partial f}{\partial t}(t_{\min},\epsilon') = 0,$$

and so the coefficient is

$$\alpha' \cos(2\sigma t_{\max} + \delta') - \alpha' \cos(2\sigma t_{\min} + \delta'),$$

and this vanishes when $\epsilon' = 0$, since $2\sigma t_{\max}$ and $2\sigma t_{\min}$ then differ by 2π. Thus the coefficient of ϵ' in $f\{t_{\max}(\epsilon'),\epsilon'\} - f\{t_{\min}(\epsilon'),\epsilon'\}$ vanishes, and the power series is

$$f_{\max} - f_{\min} = 2\alpha'\{1 + O(\epsilon')^2\},$$

as previously stated in equation (4.10).

4.4. *Non-linear effects*

The object of the present investigation is to test the validity of the linearized theory, and accordingly no correction need be made for non-linear effects. Actually, since there is as yet no non-linear wavemaker theory, these corrections could not have been included even if we had wished to do so.

5. Experimental apparatus

The wave channel in which the experiments were conducted is of rectangular cross-section with glass sides and bottom and is 3 ft. deep, $2\frac{1}{2}$ ft. wide and approximately 100 ft. long (see figure 1). A piston-type wavemaker was used to

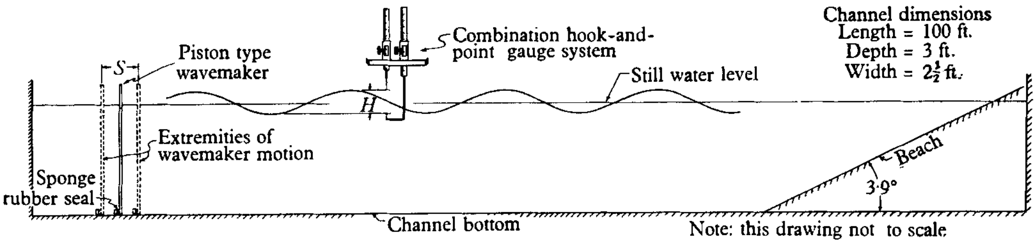

FIGURE 1. Schematic diagram of experimental equipment.

generate the waves. A beach with a slope of 1:15 at the far end of the channel from the wavemaker was used for the absorption of wave energy. A combination hook-and-point gauge was used for the measurement of the waves. These components are described in the following paragraphs.

5.1. *The wavemaker*

The wavemaker is a smooth aluminium plate oriented perpendicular to the axis of the channel and oscillates in a direction parallel to the axis of the channel. There is a small clearance of approximately $\frac{1}{4}$ in. between the edges of the wavemaker and the side-walls and bottom of the channel which was closed by an adjustable sponge rubber seal, thereby reducing the leakage around the edges of the wavemaker.

The motion of the wavemaker is controlled by a hydraulic (oil) servomechanism system. The shaft of a double-acting cylinder is connected directly to a carriage by which the wavemaker is supported. Flow of oil under pressure into and out of the two chambers of the double-acting cylinder drives the wavemaker and is metered by a valve, the output of which is prescribed by a rotating cam acting through a mechanical leverage system, magnifying the input from the cam. The linear displacement of the wavemaker with time is designed to be the same (except for a scale factor) as the angular variation of the distance from the surface of the cam to the rotational axis of the cam. By use of different cams, each with a different control surface, the motion of the wavemaker can, in principle, be given any desired periodic form, but in practice the response was imperfect. To test the theory described in § 3 the motion of the wavemaker was desired to be simple harmonic. Actually, we found at the beginning of the investigation that with a simple harmonic cam the wavemaker motion was far from simple harmonic, but consisted of short steps rather than a smooth and continuous curve. This effect was noted particularly at the lower frequencies. Tests showed that it was due to friction in the seals of the hydraulic piston and could be reduced by continuously rotating the shaft of the piston. Further improvement was sometimes achieved by trial-and-error adjustment of the valves controlling the oil pressures in the hydraulic system. The motion of the wavemaker was then nearly simple harmonic with the second harmonic not exceeding 5 % of the fundamental and the third harmonic being presumably much smaller. (In our analysis of the measurements, the second harmonic is in effect eliminated, see § 4.3 above.) To obtain a simple-harmonic wave motion, other investigators have used wave filters (e.g. Biésel 1948); however, since a wave filter involves viscous dissipation, it could not have been used in this investigation which attempts to verify an inviscid theory.

The stroke of the wavemaker can be varied from 0·01 to 2 ft. by changing the length of the lever arm which transmits the motion of the cam to the valve. The periodicity can be varied from 0·7 sec to arbitrarily long periods by changing the rotational speed of the cam. A more detailed description of the wavemaker is given by Ippen & Eagleson (1955).

5.2. *The beach*

The beach is of plane, impermeable, varnished material, with a small gap (about $\frac{1}{4}$ in.) between the beach edges and the walls of the flume through which seepage could occur. The slope of the beach is 3·9 degrees. It was found that in most cases the reflected wave heights was less than 10 % of the incident wave height (see § 7.1 below). This compares favourably with the efficiency of other known absorbers, e.g. see Greslou & Mahe (1954) and Herbich (1956).

5.3. *Combination hook-and-point gauge*

This was used for the measurement of wave height. It is an adaptation of a well-known method for the measurement of water-surface elevations which are steady (i.e. time-independent) or varying very slowly. In the final version of the gauge, the hook and point were on the same base, but could be adjusted independently. The hook and point portions of the gauge were used to indicate the trough and crest elevations, respectively. The difference between these levels can be shown to be independent of any second-harmonic component which may be present (see § 4.3).

The tip of the hook gauge was made of wire of small diameter (0·06 in.), and there was no noticeable flow disturbance due to the tip for most waves used in this test. Both the hook and point portions of the gauge were attached to graduated staffs equipped with an adjusting knob and a vernier graduated to read displacements of the staff to the nearest 0·1 mm. The gauges were mounted with their tips in a vertical line and were both attached to a movable carriage which could be rolled on top of the channel to any longitudinal position along the channel and could be clamped rigidly in position for the measurement of a wave height. A source of light, mounted on the opposite side of the channel from the observer, caused a reflected light pattern on the water surface and was useful in determining when the hook and point tip made contact with the water surface. When the hook or point gauge just touched the water surface, the curvature of the water surface at this point changed abruptly, thereby interrupting the reflected light pattern at this point. It is estimated that each gauge could be read with a maximum error of 0·1 mm. In many cases the unsteadiness of the water surface was of the same order, and to allow for it the elevation of the crest (similarly of the trough) was defined as the elevation at which the gauge made contact with half the crests (or troughs). For high waves, the higher particle velocities at the trough position caused ripples near the tip of the hook gauge and increased the uncertainty involved in the adjustment of the hook gauge elevation.

In order to obtain a measurement of wave height, it is necessary to know the difference in the two vernier readings when the tips of the two gauges are at the same elevation. This was obtained from readings at a still water surface. Since both gauges are on the same base, this difference is independent of the location of the gauge system along the channel.

At first a capacity-wire gauge was used for wave measurement, and this was estimated to have an absolute error of about 2 mm (this agrees with the estimate of Tucker & Charnock (1955)). An absolute error of this magnitude is too large for our present purposes; for the size of waves used in this study (1–5 cm) the error would range approximately from 4–20 %. This gauge is also described in detail by Ippen & Eagleson (1955).

An inherent disadvantage of the hook-and-point gauge is that only the wave height is obtained, while the capacity-wire gauge gives a continuous measurement of the water surface elevation; however, the absolute error associated with the hook-and-point gauge is much less than that associated with the capacity-wire gauge. Since we were concerned mainly with small-amplitude waves, where the error is limited by the absolute error of the gauge, we adopted the hook-and-point gauge and were content to regard the measurements of wave height (rather than of complete profiles) as a sufficient experimental check on the theory.

6. Methods of measurement

The theory of § 3 predicts a relation (equation (3.9)) between the ratio

$$\frac{\text{wave height at a distance from the wavemaker}}{\text{stroke of wavemaker}} = \frac{H}{S} \tag{6.1}$$

and the ratio

$$\frac{2\pi \times \text{water depth}}{\text{wave length}} = \frac{2\pi h}{L}. \tag{6.2}$$

It is this relation that was tested in the present set of experiments, with the aim of verifying the theory. For this purpose, it is necessary to measure the following variables: (1) water depth, (2) wavemaker stroke, (3) wavelength, (4) wave height.

6.1. *Water depth*

The water depth was measured directly (in ft.) with a wooden scale immersed through the still water surface. For the range of depths involved, the error in measured depth was less than 2 %. The resulting error in the ratio (6.1) due to a 2 % error in the ratio (6.2) is equal to or less than 2 % (actually much less for short waves).

6.2. *Wavemaker stroke*

The wavemaker stroke was measured (in cm) by fixing a pointer on the wavemaker carriage. Directly behind the pointer, a metre stick was fixed to a stationary support, and so the relative positions of the extremities of motion could be

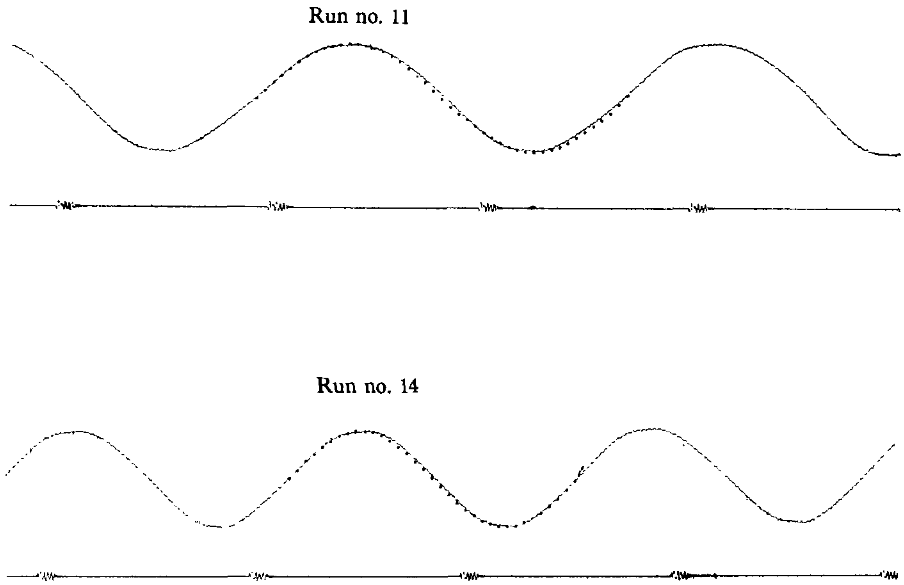

FIGURE 2. Sample of wavemaker motion (continuous line) compared with sine curve (dotted line).

measured visually. Attempts by different observers to reproduce the readings resulted in a maximum difference in the measured stroke of 0·2 mm. The resulting maximum proportional error is 2 % for the range of strokes used in this investigation. A sample of the form of the wavemaker motion was also measured for each run by using a mechanical linkage to convert the translational displacement of the wavemaker into a small angular displacement. A variable rotary capacitor converted the angular displacement into an electrical signal which was recorded by a commercial Sanborn Model 150 oscilloscope. No significant non-linear effects were introduced by this method of conversion of the translation motion into an angular displacement. (See figure 2 for a comparison of measured wavemaker motion with simple harmonic motion.)

7. Measurement of effects not included in the simple theory

The present section describes measurements of effects not included in the simple wavemaker theory, some of which have been discussed in §4: (1) the incident wave is partially reflected from the sloping beach, see §4.1; (2) the wave height is attenuated because of viscous dissipation, see §4.2 where a theoretical expression for the attenuation coefficient was presented; (3) the waves may not be strictly two-dimensional; there may be transverse reflexions and transverse slopes of the water surface.

7.1. *Measurement of reflexion from the beach*

As explained in § 4.1, partial reflexion of the incident wave from the beach causes a partial standing wave system within the channel such that the wave height is not the same at all locations along the channel, but oscillates about

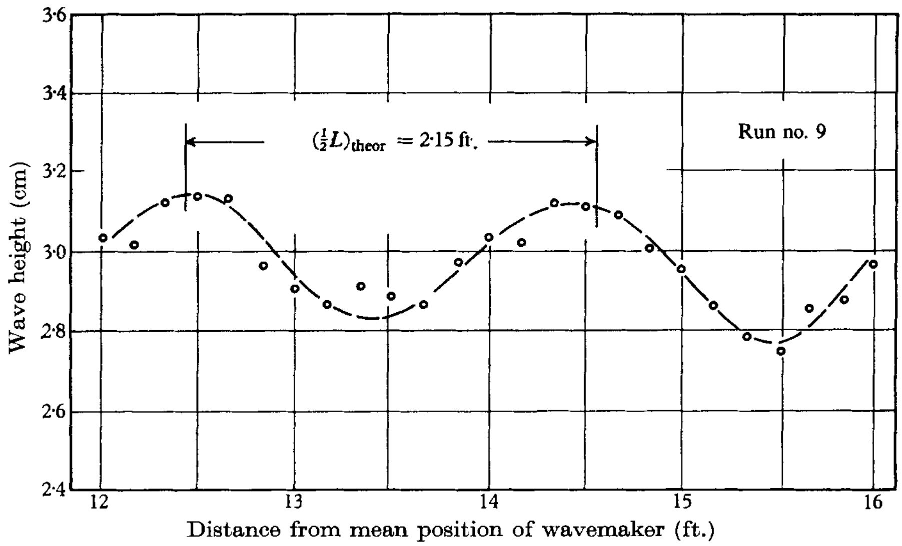

FIGURE 3. Measurement of wave height against distance. (Wavelength calculated from frequency by small-amplitude theory.)

a mean value, the wavelength of the oscillation being $\frac{1}{2}L$. Figure 2 shows one experimental measurement of wave-height variation with distance, showing the expected reflexion effect. If the maximum and minimum values of the wave-height envelope (figure 3) are denoted by H_{max} and H_{min}, respectively, then the amplitude reflexion coefficient of the beach ϵ_r defined as the ratio of the reflected wave height to the incident wave height, may be expressed in the form (see also equation (4.4)

$$\epsilon_r = \frac{H_r}{H_i} = \frac{H_{\text{max}} - H_{\text{min}}}{H_{\text{max}} + H_{\text{min}}}. \tag{7.1}$$

Reflexion coefficients were calculated from the measurements wherever the variation of wave height with distance was of the form to be expected from the simple reflexion theory. In several cases, the wave height variation with distance was not of wavelength $\frac{1}{2}L$, and there is some evidence that this may be due to transverse effects (see § 7.3 below). The reflexion coefficients are plotted against deep-water wave steepness, H_0/L_0 in figure 4, and are seen to be mostly of the order of 10 % or less. The mechanism of wave reflexion by a sloping beach is not yet understood. A formula for the reflexion coefficient has been given by Miche (1951) which seems to predict the right order of magnitude, although it does not claim to represent the physical processes correctly. Miche's formula is also shown in figure 4. Miche first considers a standing wave on an impermeable beach of constant small slope β. The slope of the water surface at the shore line varies between a maximum (positive) and a minimum (negative) value which are (according to the linear theory) proportional to the deep-water steepness H_0/L_0. Miche defines a critical deep-water standing-wave steepness $(2H_0/L_0)_{\text{crit}}$, for which the maximum water surface slope at the beach is formally equal to the

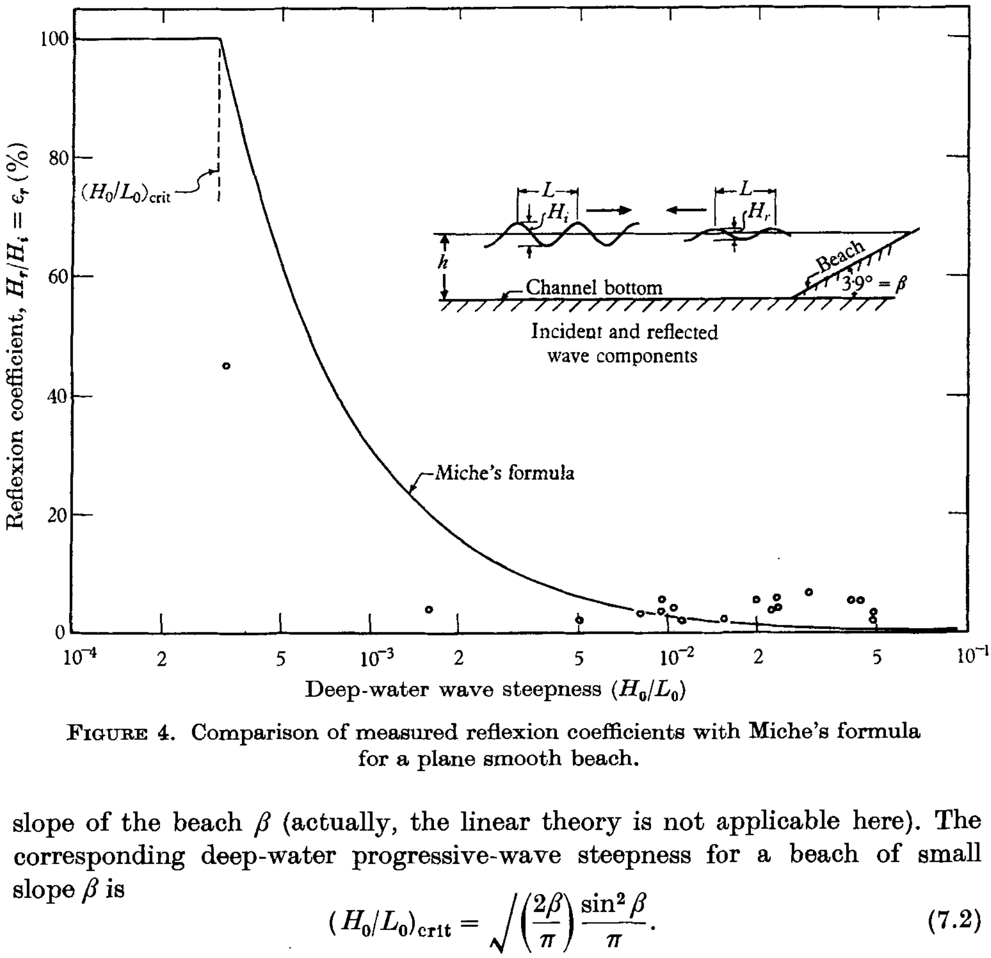

FIGURE 4. Comparison of measured reflexion coefficients with Miche's formula for a plane smooth beach.

slope of the beach β (actually, the linear theory is not applicable here). The corresponding deep-water progressive-wave steepness for a beach of small slope β is

$$(H_0/L_0)_{\text{crit}} = \sqrt{\left(\frac{2\beta}{\pi}\right)}\frac{\sin^2\beta}{\pi}. \tag{7.2}$$

Miche further suggests that the reflexion coefficient ϵ_r, the progressive wave deep-water steepness H_0/L_0, and the *critical* progressive wave deep-water steepness $(H_0/L_0)_{\text{crit}}$ are related by

$$\epsilon_r = 1 \quad \text{for} \quad H_0/L_0 \leqslant (H_0/L_0)_{\text{crit}}, \tag{7.3}$$

$$\epsilon_r = \frac{(H_0/L_0)_{\text{crit}}}{H_0/L_0} \quad \text{for} \quad H_0/L_0 \geqslant (H_0/L_0)_{\text{crit}}. \tag{7.4}$$

Miche also modifies his formulas with an empirical coefficient to take into account the effect of roughness. While the arguments leading to equations (7.3) and (7.4) seem to lack a real physical basis, the predictions agree quite well (at least in order of magnitude) with the measurements; see also Herbich (1956) for other measurements of reflexion.

7.2. *Attenuation*

Assume that the reduction in wave height (mainly due to viscous effects) as the wave progresses down the channel can be described by

$$H_2 = H_1 e^{-K'(x_2-x_1)},$$

where H_2 and H_1 are the wave heights at stations x_2 and x_1 respectively, and K' is the 'attenuation coefficient' with the dimensions of reciprocal length. A theoretical expression for the attenuation coefficient was presented in § 4.2 (equation (4.8)). Two measurements of the attenuation coefficient were made by measuring the wave height (averaged over a half wavelength) at two stations

Run no.	H_1 (Station 14)* (cm)	H_2 (Station 51) (cm)	h (ft.)	T (sec)	L (ft.)	K'_{meas} (ft.$^{-1}$)	K'_{theor} (ft.$^{-1}$)
17	2·20	2·17	1·67	1·273	7·38	0·00038	0·00064
18	2·62	2·55	2·31	0·927	4·39	0·00070	0·00098

* Note. Stations are measured (in feet) from the mean position of the wavemaker with: $b = 2{\cdot}5$ ft., and $\nu = 1 \times 10^{-5}$ ft.^{2}/sec.

TABLE 1. Comparison of measured and theoretical attenuation coefficients

37 ft. apart along the channel (see Table 1), and it was found that the decrease in wave height over the constant-depth portion of the channel was so small (about 0·5 mm) as to be comparable with the absolute error. The conclusion is that, if the wave height is measured within 10 or 20 ft. of the wavemaker, the wave can be considered unattenuated for practical purposes. The wave characteristics and the values of the measured and theoretical attenuation coefficients are shown in Table 1.

The measured and theoretical attenuation coefficients are in better agreement than could have been expected when the small difference in wave height between the two stations is considered; particularly since the probable error in the measurement of H_1 and H_2 is of the same order as the difference of H_1 and H_2. It is interesting to note that some investigators (e.g. Benjamin & Ursell 1954, and Case & Parkinson 1957) had previously found measured attenuation coefficients which were greater than predicted by theory. One possible explanation for measured attenuation coefficients greater than theoretical predictions is that experiments performed on a small scale may include surface dissipation effects not included in the theory.

7.3. *Three-dimensional variations*

If all the generative and dissipative forces acting on the wave system were uniform across the channel, then no variation of wave height along the transverse dimension of the channel would be expected. In some cases effects were observed which may be due to the following three-dimensional causes: (1) Symmetric or asymmetric seepage past the edges of the wavemaker, or similar seepage between the edges of the beach and the side-walls of the channel. (2) Dissipation along the walls of the channel. (3) Breaking, or backflow after breaking, of a wave which is not uniform across the channel. (4) Three-dimensional instabilities (see Schuler 1933). These effects were usually too small to be measurable; however, in some instances the variation of wave height with distance was not even approximately of the simple form (wave length $\frac{1}{2}L$) predicted in § 4.1, but wavelengths of $\frac{1}{4}L$ and $\frac{1}{2}L$ were also prominent. When this occurred, the wave motion was seen to be three-dimensional, although the reasons are not understood. A measurement of wave height *across* the channel for one of these cases showed the wave height varying approximately antisymmetrically about the centre of the channel. The difference in wave height at the two sides of the channel was of the order of 1 mm. Simple averaging of the wave height over a distance of $\frac{1}{2}L$ on the centre line of the channel was used to reduce three-dimensional effects, as well as reflexion effects. This averaging process also tends to reduce any higher harmonics in the wave height which may be introduced by higher harmonics in the motion of the wavemaker.

8. Test of wavemaker theory; discussion and conclusions

In order to test the theory for a piston wavemaker, the measured wave height was compared with the theoretical wave height as predicted by equation (3.9). In order to check the theory, it is necessary to measure wave height H, wave period $2\pi/\sigma$, water depth h, and wavemaker stroke S. Of these, the wave height cannot be found from a single measurement; the wave height was measured over a distance of at least $\frac{1}{2}L$, and the primary incident wave height, $H = 2\alpha$, was deduced by the analysis described in § 4.1. (There were six runs where the height variation was not even approximately of the form given by equation (4.3); see the discussion of § 7.3. In these cases, only a crude average over $\frac{1}{2}L$ was taken.) From the wave period $2\pi/\sigma$, the wavelength $2\pi/k_0$ was calculated by use of the formula $\sigma^2 = gk_0 \tanh k_0 h$; then the ratios H/S and $k_0 h$ were calculated. According to the theory, they should be related by equation (3.9).

The measurements were first made for small wave steepnesses

$$(0{\cdot}002 \leqslant H/L \leqslant 0{\cdot}03)$$

to test the validity of the small amplitude theory. The comparison with theory is shown in figure 5.

Each experimental point assigns a value of H/S to a value of $k_0 h$; for the same value of $k_0 h$ a theoretical value of H/S can be calculated, and the difference between this and the experimental value may be defined as the deviation between theory and experiment. Thus, for every point a percentage deviation is obtained. The deviation for the small wave steepnesses is found to be 5 %; however, much of this comes from a single measurement for which the percentage deviation is 32 % and the reflexion coefficient is 45 % (which is in any case too high for the correction of § 4.1 to be applicable). If this measurement is henceforth neglected, the average percentage deviation is 3·4 % below the theoretical curve. The scatter about a mean curve drawn 3·4 % below the theoretical curve is of the order of 3 %, and this is a reasonable estimate for the experimental error since the errors in measurement of wave height, wavelength, water depth, and wavemaker stroke have each been estimated to be of the order of 2 %. Another source of scatter should be mentioned at this point. According to equation (4.3), the variation of wave height with distance should be sinusoidal. Actually, it is found that the envelope though smooth is not exactly of the predicted form. In particular,

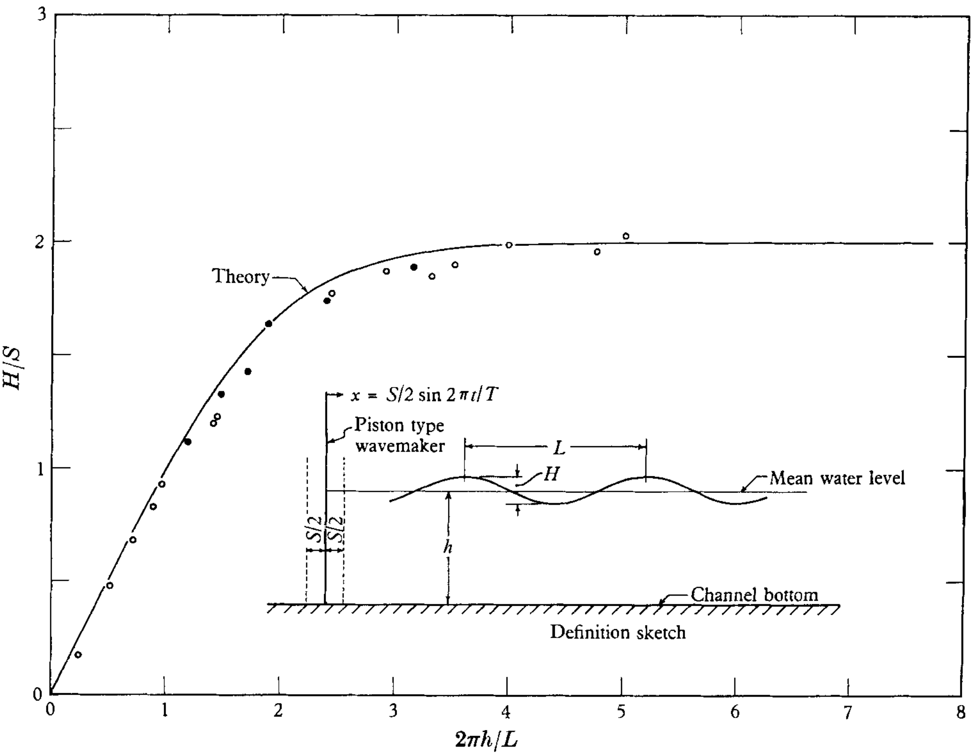

FIGURE 5. Test of wavemaker theory for small wave-steepnesses. o, Experiments corrected for reflexion; •, experiments not corrected for reflexion.

H_{max} and H_{min} do not repeat exactly at successive maxima and minima of the wave-height envelope. If the analysis of § 4.1 is, nevertheless, applied, there is an uncertainty in the wave height, leading to a scatter. If the mean uncertainty is defined as one-half the average of the differences of successive maxima plus one-half the average of the differences of successive minima divided by the average wave height, the mean percentage uncertainty is found to be of the order of 3 %. The reason for H_{max} and H_{min} not exactly repeating is not understood, although it is felt that it may be a result of interference with transverse reflexions.

The deviation between theory and experiment is measured by the systematic mean deviation of 3·4 %. This small deviation between theory and experiment may be partly explained by two effects, each tending to cause deviations in the observed direction. These are (i) leakage past the wavemaker, and (ii) finite-amplitude effects (see the next paragraph). Further improvements in equipment, measuring technique and analysis would probably lead to even better agreement with theory, for sufficiently low waves.

To test the limitation which finite-amplitude effects place on the range of validity of the small-amplitude theory, measurements were also made on steeper waves ($0{\cdot}045 \leqslant H/L \leqslant 0{\cdot}048$). The comparison of these measurements with small amplitude theory is shown in figure 6. The mean deviation is 10 % below the theoretical curve; the mean scatter of the experimental results about a curve 10 % below the theoretical curve is again of the order of 3 %. The systematic deviation of 10 % is evidence that the effect of finite amplitude is to cause the

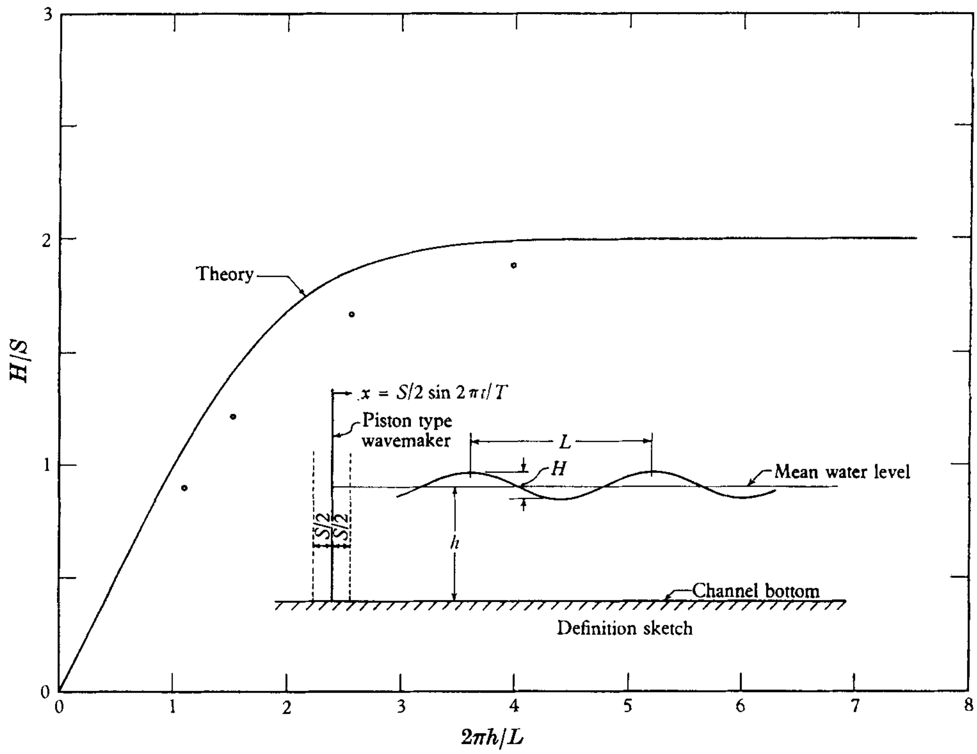

FIGURE 6. Deviation from wavemaker theory due to finite wave steepness. (All points are corrected for reflexion.)

wave height to be less than predicted by small amplitude theory. The larger reduction, however, observed in the Neyrpic experiments can only be partly attributed to finite-amplitude effects.

Two measurements of attenuation coefficients were made, and these were found to agree in order of magnitude with the theoretical predictions. In order to check the theory (based on a laminar boundary layer) with greater accuracy, it would be necessary to have a channel with a larger ratio of wetted area to cross-sectional area; at the same time the size of the equipment must be large enough to avoid surface tension effects.

The main limitation to experimental accuracy lies in the lack of a really efficient energy absorber. This study utilized a beach of small slope (3·9°) with a resulting amplitude reflexion coefficient less than 10 % in most cases. The reflexion was allowed for in the analysis; however, the possible error would be reduced and the analysis simplified if a more efficient absorber could be found.

No explanation can be given for the rather large systematic deviation between theory and experiments found in the earlier Neyrpic experiment on a paddle wavemaker. The systematic deviation found in the present set of experiments on

no. of run	H (cm)	S (cm)	h (ft.)	T (sec)	$\frac{2\pi h}{L}$	$(H/S)_{meas}$	$(H/S)_{theor}$	ϵ_r (%)	(H_0/L_0)	Uncertainty in H/S (%)
1	2·67	5·66	2·01	3·23	0·51	0·48	0·51	4·1	0·00157	+
2	4·07	4·93	2·01	1·68	0·89	0·83	0·88	4·2	0·0103	+
3*	3·63	2·22	2·01	1·17	1·89	1·64	1·62	—	0·0170	—
4	2·56	1·37	2·01	0·92	2·92	1·87	1·92	2·1	0·0110	+
5*	1·77	1·34	2·19	1·42	1·49	1·33	1·37	—	0·00608	—
6	2·13	1·05	2·40	0·77	5·01	2·03	2·00	4·3	0·0230	2
7	2·01	1·03	2·40	0·79	4·76	1·96	2·00	3·9	0·0220	7
8	3·33	1·67	2·40	0·86	3·99	1·99	1·99	6·9	0·0293	6
9	2·88	1·51	2·40	0·92	3·52	1·90	1·97	5·9	0·0230	3
10*	2·71	1·56	2·40	1·12	2·40	1·74	1·82	—	0·0142	—
11*	2·03	1·82	2·40	1·73	1·19	1·12	1·15	—	0·00472	—
12*	2·40	1·68	2·40	1·36	1·71	1·43	1·52	—	0·00887	—
13	2·77	1·56	2·40	1·11	2·44	1·77	1·82	2·4	0·0153	0
14	2·10	2·25	1·58	1·63	0·97	0·93	0·97	2·1	0·00500	5
15	1·40	2·06	1·57	2·09	0·72	0·68	0·70	5·7	0·00957	1
16	0·81	4·80	0·63	3·67	0·24	0·17	0·25	45·1	0·00326	20
17	2·26	1·88	1·67	1·27	1·42	1·20	1·32	3·6	0·00944	2
18	2·72	1·45	2·31	0·93	3·31	1·85	1·96	5·7	0·0197	3
19	1·84	1·50	1·67	1·25	1·45	1·23	1·33	3·3	0·00799	1
20*	2·00	1·06	2·28	0·94	3·16	1·89	1·94	—	0·0145	—
21	4·77	2·54	2·00	0·79	3·98	1·88	1·99	2·2	0·0488	1
22	5·25	3·15	1·50	0·85	2·55	1·67	1·85	3·6	0·0485	3
23	5·47	4·50	1·00	0·95	1·51	1·22	1·39	5·4	0·0439	2
24	5·14	5·73	0·66	0·96	1·09	0·90	1·05	5·5	0·0409	2

Note:

(*a*) Runs 1 to 20 are for low wave-steepnesses ($0{\cdot}002 \leqslant H/L \leqslant 0{\cdot}03$).

(*b*) Runs 21 through 24 are for high wave-steepnesses ($0{\cdot}045 \leqslant H/L \leqslant 0{\cdot}048$).

(*c*) Values of H and H/S are corrected for reflexion effects except in those runs marked with superscript *, where the wave height envelope was not of the expected form. Values of ϵ_r and the 'uncertainty' were not calculated for these runs.

(*d*) Runs marked * are runs where only one maximum and minimum was obtained in the wave height envelope; therefore, the uncertainty could not be calculated.

Table 2. Summary of results

a piston wavemaker is much less. It is felt that the theory is also correct for paddle wavemakers (for small amplitude waves) and that the deviation found by the Neyrpic group can only be explained as the result of reflexion or of effects not accounted for in the theory (i.e. possibly leakage about the edges of the wavemaker, motion of the wavemaker which was not simple harmonic, or other effects).

The experiments constitute a close verification of the small-amplitude wave theory as applied to the simple case of a piston wavemaker, and provide evidence, for the first time, that this theory may be used with confidence to calculate wave amplitudes (and probably also forces) in more complex cases where experimental verification is not available.

A summary of all measurements and calculations, for the present set of experiments on a piston wavemaker, is presented in Table 2.

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